Linear dynamics of charged particles in the main lattices of storage rings

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ABSTRACT

To study the characteristics of synchrotron radiation in magnetic fields of accelerators first the author was necessary to obtain a continuous solutions of Hill's equation. For this purpose the gradient or the components of magnetic field were developed in a series. The same procedure is followed now in the case of storage rings. This approach proved to be interesting not only from the point of view of describing the motion of particles in ordinary three-dimensional space but also in the fact that we get new differential equations. This brief review can be ragarded as an introduction to the proposed approach. The next step may be to add nonlinearities. This would be the best approximation to the determination of betatron oscillations in the existing accelerators.

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In the beginning, we want to note that the asymptotics found in this manner helped to reveal the experimentally observed dependence for the spectral and angular formulas of radiation on the vertical betatron oscillations [1,2].

There are three main types of lattice for storage rings [3,4]. Among them are the FODO, the Chasman-Green lattice(DBA), and triplet achromat(TBA). The FODO lattice is the most used. Here one cell is formed of separated focusing(F) and defocusing(D) quadrupoles, bending magnets(O). Let quadrupoles have a length a and magnets are length d. Then path of single period is $L = 2a + 2d + 4l$, where l is the extension of free shifts. The closed trajectory has scale $S = 2\pi R + (2a + 4l)N$, where R is the radius of dipole magnets and N is the number of periods. Let average radius is defined as

$$2\pi R_0 = 2\pi R + 2(a + 2l)N,$$

where $2\pi R = 2dN$. This can be written as $R_0 = (1 + k)R$, where $k = (a + 2l)/d$.

The magnetic field of dipole is $B_z = B$ and components of quadrupole define as

$$H_z^f = -gx, H_z^d = gx, H_x^f = -gz, H_x^d = gz,$$

where g is the lens constant, index f and d means focusing and defocusing. x coordinate coincides with the radial direction.

The devices of one period give approximately for an axial part of magnetic field the following alternation:

$$-gx, \quad \varphi \in [0, aT];$$

$$B, \quad \varphi \in [(a + l)T, (a + l + d)T];$$

$$gx, \quad \varphi \in [\pi/N, (2a + 2l + d)T];$$

$$B, \varphi \in [(2a + 3l + d)T, (2a + 3l + 2d)T],$$

where $T = 2\pi/(NL)$, φ is the azimuth angle.

The Fourier series expansion of this magnetic fields takes the form

$$H_z = \frac{2d}{L}B + \sum_{k=1}^{\infty}[\frac{4B}{\pi}\frac{(-1)^k}{k}\cos\frac{\pi k}{2}\sin\frac{\pi k}{L}d-$$

$$\frac{2}{\pi}gx\frac{1-(-1)^k}{k}\sin k\tau_1]\cos k(\tau - \tau_1),$$

where $\tau = N\varphi, \tau_1 = \pi a/L$. In particular, in the study of a radiation problem the dipole magnetic field can be averaged and then

$$\bar{H}_z = \frac{2d}{L}B - gxf(\tau),$$

where $f(\tau) = (4/\pi)n(\tau)$,

$$n(\tau) = \sum_{\nu=0}^{\infty} f_{2\nu+1}\cos(2\nu+1)(\tau-\tau_1), \qquad f_{2\nu+1} = \frac{\sin(2\nu+1)\tau_1}{2\nu+1}.$$

If $n(\tau)$ is differentiated a divergent series is arised. The second component of field is derived as $H_r = -gzf(\tau)$.

Based on previous observations, we determine the angular velocity in the following form:

$$\dot{\varphi} = \frac{\omega_0}{1+k}(1 - \frac{x}{R_0} + \frac{3}{2}\frac{x^2}{R_0^2})+ \\ \frac{\omega_q}{R_0^2}\int f(\tau)(z\dot{z} - x\dot{x})dt, \tag{1}$$

where

$$\omega_0 = \frac{e_0 B}{m_0 c}, \quad \omega_q = \frac{e_0 g R_0}{m_0 c}, \quad 2d/L = 1/(1+k).$$

In linear approximation the equations of betatron oscillations become

$$\frac{d^2 z}{d\tau^2} + \frac{1}{N^2}\frac{(1+k)\omega_q}{\omega_0}f(\tau)z = 0, \tag{2}$$

$$\frac{d^2 x}{d\tau^2} + \frac{1}{N^2}[1 - \frac{(1+k)\omega_q}{\omega_0}f(\tau)]x = 0. \tag{3}$$

Let us introduce new constants

$$C_1 = \frac{gR_0(1+k)}{B}, \qquad \lambda^2 = \frac{4C_1}{\pi N^2}.$$

Under the existing g and B it turns out that parameter $\lambda >> 1$. Eq. (2) can be rewritten as

$$\frac{d^2 z}{d\tau^2} + \lambda^2 n(\tau)z = 0. \tag{4}$$

Expression (4) is the Hill equation with a large parameter. It is also differential equation with periodic coefficient and with a small parameter at the

highest derivative. By way of illustration several procedures of solution for Eq.(4) will be tested.

To find the first solution we can take the WKB-method [5]. Let us put

$$z = exp(i\lambda G(\lambda,\tau)) \cdot \varphi(\tau),$$

where $G(\lambda,\tau) = G_0(\tau) + G_1(\tau)/\lambda + ...$ In this case instead of Eq.(4) we have

$$\frac{d^2\varphi}{d\tau^2} + 2i\lambda\frac{dG}{d\tau}\frac{d\varphi}{d\tau} + [i\lambda\frac{d^2G}{d\tau^2} - \lambda^2(\frac{dG}{d\tau})^2 + \lambda^2 n(\tau)]\varphi = 0.$$

Bearing in mind here terms with λ^2, we obtain $G_0 = \int \sqrt{n(\tau)}d\tau$. The following approximation gives $\varphi = 1/\sqrt[4]{n(\tau)}$. Thus, solution is

$$z = (1/\sqrt[4]{n(\tau)}) \cdot \exp(i\lambda \int \sqrt{n(\tau)}d\tau) + ...$$

But the solution may only depend on the double integral $\iint n(\tau)d\tau d\tau$.

According to [5] again let us use substitution $u = \lambda^2\tau$ which changes Eq.(4) as

$$\frac{d^2z}{du^2} + \frac{1}{\lambda^2} \cdot n(\frac{u}{\lambda^2})z = 0.$$

It may be formally solved as an equation with a small parameter. Assume that the solution has the form

$$z = \exp(i\gamma\, u) \cdot \varphi(u),$$

where

$$\varphi(u) = \varphi_0 + \frac{1}{\lambda^2}\varphi_1 + \frac{1}{\lambda^4}\varphi_2 + ..., \quad \gamma = \frac{\gamma_1}{\lambda^2} + \frac{\gamma_2}{\lambda^4} + ...$$

Note that in the region of stability the value of γ must be real. At various powers of λ we shall obtain $\varphi_0 = C$, where C is the constant, $\varphi_1 = C\lambda^4 C_{21}$, where

$$C_{nk} = \sum_{\nu=0}^{\infty} \frac{f_{2\nu+1}}{(2\nu+1)^n} \cos k(2\nu+1)\tau_c$$

with $\tau_c = u/\lambda^2 - \tau_1$. Besides, eliminating the secular terms we find

$$\gamma_1^2 = \frac{1}{2}\lambda^4 \sum_{\nu=0}^{\infty} \frac{f_{2\nu+1}^2}{(2\nu+1)^2} \quad \text{and} \quad \gamma \approx \frac{\pi^2 a}{4L}\sqrt{1 - \frac{4a}{3L}}.$$

Asymptotics with an additional term φ_2 takes the form

$$z = C\exp(i\frac{\gamma_1}{\lambda^2}u)[1+\lambda^2 C_{21} - 2i\gamma_1\lambda^2 S_3 + \frac{1}{8}\lambda^4 C_{42} + \frac{1}{8}\lambda^4 S_{\mu\nu}], \tag{5}$$

where

$$S_{\mu\nu} = \sum_{\mu=0}^{\infty} \frac{f_{2\mu+1}}{(2\mu+1)^2} \sum_{\nu=0}^{\infty} f_{2\nu+1}[\frac{\cos 2(\mu-\nu)\tau_c}{(\mu-\nu)^2} + \frac{\cos 2(\mu+\nu+1)\tau_c}{(\mu+\nu+1)^2}],$$

$$S_n = \sum_{\nu=0}^{\infty} \frac{f_{2\nu+1}}{(2\nu+1)^4} \sin(2\nu+1)\tau_c, \qquad \mu \neq \nu.$$

Furthermore solution (5) is reduced to the superposition of sine and cosine with extended modulated amplitudes. Note that a real part of terms in braces is coefficient of cosine. In (5) function $n(\tau)$ is absent. This technique as it is able to derive the frequency of prevalent oscillations $\nu_z \sim \lambda^2 N\gamma$ but amplitudes will be increased.

The close equation was also examined in [6]. Keeping in mind about the double integration of $n(\tau)$ let us introduce by analogy to [6] new variables:

$$p(t) = \iint n d\tau d\tau, \quad x = \int \sqrt{p(\tau)}d\tau, \quad v = \sqrt[4]{p(\tau)} \cdot z.$$

After transformation Eq.(4) becomes

$$\frac{d^2 v}{dx^2} + \lambda^2 v = \frac{1}{16}\frac{v}{p^3}(4np - 5p'^2).$$

Here the left part has not a periodic coefficient but the solution of inhomogeneous equation will be intricate.

Let us pass on to the Chasman-Green lattice. In this case there is in centre focusing quadrupole of length a_1. Then the cell contains, on both sides one after another straight sections with lengths l_1, l,l_2, between bending magnets of length d, defocusing and focusing quarupoles of length a. For one period L is

$$2d + 4a + a_1 + 4l + 2l_1 + 2l_2.$$

Corresponding equation for the vertical oscillations can be expressed in the form

$$\frac{d^2 z}{d\tau^2} + \frac{C_1}{N^2}(\frac{a_1}{L} + \frac{2}{\pi}\sum_{\nu=1}^{\infty}\frac{f_1}{\nu}\cos\nu\tau)z = 0, \tag{6}$$

where $k = (L - 2d)/2d$,

$$f_1 = 4 \sin \tau_2 a \sin \tau_2 (a + l) \sin \tau_2 (2a + l + 2l_2) +$$

$$(-1)^\nu \sin \tau_2 a_1, \tau_2 = \pi\nu/L.$$

Contrary to the Eq.(4), given relation contains an additional constant part along with the trigonometric series.

An equation for triplet achromat cell is close in form to Eq.(6). Here there is in centre the defocusing quadrupole of length a_1, then laterally through the free intervals are focusing quadrupole of length a and bending magnet of length d. For example, path of right side is equal to

$$a_1 + l_1 + a + l_2 + d + l_3,$$

where l_i is the length of straight sections.

In linear case the equation of axial oscillations becomes

$$\frac{d^2 z}{d\tau^2} + \frac{C_2}{N^2}[\frac{2a - a_1}{L} + \frac{2}{\pi}\sum_{\nu=1}^{\infty}\frac{(-1)^\nu}{\nu} f_2 \cos \nu\tau]z = 0, \tag{7}$$

where $C_2 = \omega_q(1 + k)/\omega_0$,

$$f_2 = 2 \sin \tau_2 a \cos \tau_2 (2l_1 + a + a_1) - \sin \tau_2 a_1.$$

Formulas (4)-(7) are unusual differential equations. In particulare, we cannot differentiate periodic coefficients, as will be broken convergence of the series. Hill's method also could not be used because the infinite determinant increases for $\lambda >> 1$. This problem has a boundary layer, as quadrupoles operate in very narrow zones and are responsible for the emergence of a small parameter at the highest derivative. But the derived equations allow to perform simulation. Taking into account the injection of particles and the state of the beam at specific points, we can introduce the initial conditions and to solve the Cauchy problem. Methods of this article can be used to study the motion of particles in more complex magnetic systems including sextupoles and wigglers.

REFERENCES

[1] O.E. Shishanin, Journ.of Exper.and Teor.Phys. **90**, 725(2000).

[2] O.E. Shishanin, Nucl.Inst.and Meth.in Phys.Res. A **558**, 74(2006).

[3] H. Wiedemann, Nucl.Inst.and Meth.in Phys.Res. A**246**, 4(1986).

[4] H. Wiedemann, Particle Accelerator Physics. Springer, 2007.

[5] A.H. Nayfen, Introduction to Perturbation Techniques. John Wiley & Sons, 1981.

[6] A. Erdelyi, Asymptotic Expansions. Dover Books on Mathematics, 2010.